

Phase transitions in heavy-quark QCD from an effective theory

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Abstract. With combined hopping parameter and strong coupling expansions, we calculate a dimensionally reduced Polyakov-loop effective theory valid for heavy quarks at nonzero temperature and arbitrary chemical potential. We numerically compute the critical endpoint of the deconfinement transition as a function of quark masses and number of flavours. We also investigate the applicability of the model to the low- T and high density region, specifically in terms of baryon condensation phenomena.

1. Introduction

Despite its brilliant achievements in tackling perturbative problems, analytical investigations of QCD can not do much when it comes to determining the structure of its phase diagram in the (T, μ) plane, with T temperature and μ quark chemical potential. In nonperturbative regimes, the answers numerical lattice QCD calculations can provide are mainly limited to $\mu \simeq 0$: the notorious sign problem renders the traditional Monte Carlo sampling approach meaningless and one has to rely on alternative methods, whose efficiency typically degrades about $\mu \sim T$: examples are using an imaginary chemical potential, performing a Taylor expansion around $\mu = 0$ and reweighting of ensembles generated at zero chemical potential.

Yet, the relevant physics for heavy-ion collisions, as well as for compact-star astrophysics, takes place at far-from-zero μ : it would then be desirable to devise strategies able somehow to circumvent the problems. In this category fall the many effective approaches developed throughout the years, such as sigma models, (p)NJL models and so on.

In particular, in characterising the $T \rightarrow 0$ region of the (T, μ) phase diagram, the phenomenon of *Silver Blaze* is expected based on very general considerations: there, the partition function – hence all interesting observables – have to be independent of the chemical potential, up to a critical value associated to the lightest baryonic state in the theory. In the path-integral representation of QCD, the Dirac eigenvalues display an explicit dependence on μ , therefore the Silver Blaze phenomenon requires a highly non-trivial cancellation between them. Due to the shortcomings of the numerical methods mentioned above, the phenomenon has not been reproduced successfully in lattice QCD so far; also on the analytical side, an explicit proof exists

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only for the choice of an isospin chemical potential $\mu_I = \mu_u = -\mu_d$, with a critical onset located at $\mu_I = m_\pi/2$ [1].

It is perhaps possible to avoid the sign problem altogether, thus exploring numerically the cold dense region $\mu \ll T$ of QCD, by employing the so-called Langevin dynamics [2]: recently, numerical results for a scalar complex field have been obtained [3] and later reproduced in the context of a flux representation for the same theory [4]; both techniques, however, are at the moment hardly applicable to QCD, and it is not at all guaranteed that they will ever be.

In this work we present an effective, dimensionally-reduced Polyakov-loop model obtained by applying systematically strong-coupling and hopping-parameter expansions to the original QCD lattice formulation, valid as long as quarks are not too light and the gauge coupling β is not too large; after briefly describing the theory (Section 2), we apply it to the cold dense region (Section 3) and show that the signature of the Silver Blaze phenomenon can be read off the numerical results. A more detailed presentation of the theory can be found in [5, 6]; for the Silver Blaze property in this context, see also [7].

2. The effective theory

The model lives on a three-dimensional lattice, with complex scalars as per-site degrees of freedom, representing the traced Polyakov loops $L_x \equiv \text{Tr}(W_x)$, $W_x \in SU(3)$. The associated effective action consists of a pure-gauge term and a fermionic contribution; the former comes from application of strong-coupling methods to the original four-dimensional Yang-Mills gauge action, and the latter is obtained with a hopping-parameter expansion, starting from the standard Wilson expression for the fermionic term in lattice QCD. In general, then, we have

$$Z_{\text{eff}}(\dots) = \int \underbrace{\left(\prod_x dL_x e^{V(L_x)} \right)}_{[\mathcal{D}L]} e^{-S_{\text{eff}}[L]} , \quad \int dL e^{V(L)} = \int_{-\pi}^{+\pi} d\theta \int_{-\pi}^{+\pi} d\phi e^{2V(L(\theta, \phi))} , \quad (1)$$

where the parametrisation of the scalar L is made explicit (see [5]). The partition function Z depends on effective couplings which are in turn some functions of the original couplings appearing in the four-dimensional lattice QCD action.

Let us now sketch how the pure-gauge part of the effective action is obtained [5] starting from the four-dimensional Euclidean Yang-Mills partition function at finite temperature (with time extent N_τ lattice spacings)

$$Z_{SU(3), 4d} = \int [\mathcal{D}U_\ell] \exp \left\{ \frac{\beta}{2N} \sum_p (\text{Tr} U_p + \text{Tr} U_p^\dagger) \right\} , \quad \beta = \frac{2N}{g^2} = 18u + \dots . \quad (2)$$

By integrating out the spatial degrees of freedom and subsequently applying a strong-coupling expansion, one gets a tower of interactions in the effective theory, the leading one of which is simply a “spin-spin” term connecting nearest neighbours in the fundamental representation (the superscript “ s ” denotes the compliance of the terms to the centre symmetry requirement):

$$-S_{\text{eff}}^s = \lambda_1 S_1^s + \lambda_2 S_2^s \dots ; \quad \lambda_1 S_1^s = \lambda_1(\beta, N_\tau) \sum_{\langle ij \rangle} (L_i L_j^* + L_i^* L_j) ; \quad \lambda_{i>1} = o(\lambda_1) \text{ for } \beta \rightarrow 0 . \quad (3)$$

The precise form of the map $\lambda_1(\beta, N_\tau)$ comes from an order-by-order enumeration of strong-coupling graphs having the two Polyakov lines as boundary: we calculated the series up to order u^{10} [5].⁴ Moreover, one can resum higher powers of (certain classes of) graphs and improve the

⁴ When fermions are introduced, a shift in the gauge maps between couplings is induced [6].

effective theory to a “nonlinear” form

$$Z_{\text{eff}} = \int [\mathcal{D}L] \prod_{\langle ij \rangle} (1 + 2\lambda_1 \text{Re} L_i L_j^*) ; \quad (4)$$

the important point is that this model reproduces the deconfinement transition of the original theory, Eq. (2), and once the critical point (that still falls within the range of applicability of the strong-coupling methods) is known, it can be translated back to a table $\beta_c(N_\tau)$ with a sufficient precision and in a wide enough range of N_τ to allow for a sensible continuum extrapolation of the physical deconfinement point T_c [5].

The next step is the inclusion of fermions in the theory: we perform an expansion in the hopping parameter κ , therefore the masses (we consider N_f degenerate quarks of mass $M = (1 - 8\kappa)/(2a\kappa)$) must be large enough to guarantee convergence. The Wilson action for quarks is then written as

$$-S_q = -N_f \sum_{\ell=1}^{\infty} \frac{\kappa^\ell}{\ell} \text{Tr}(H[U]^\ell) \quad (5)$$

with H is the hopping matrix. Its structure is such that the non-zero contributions after integration of the gauge fields are given by various kinds of closed loops of length ℓ ; each of their links carries a factor κ (and, if the chemical potential is turned on, an additional factor $e^{\pm a\mu}$ in the temporal direction): the expression above is then also an expansion in powers of κ . The general form of the fermion contribution to the effective action is analogous to an external field in a spin system (here a denotes the explicit breaking of centre symmetry):⁵

$$-S_{\text{eff}}^a = -2N_f \sum_{i=1}^{\infty} \left(h_i(u, \kappa, \mu, N_\tau) S_i^a + \bar{h}_i(u, \kappa, \mu, N_\tau) S_i^{a,\dagger} \right) ; \quad \bar{h}_i(-\mu) = h_i(+\mu) , \quad (6)$$

with the i -order reflecting increasingly subdominant terms. A closer inspection of the leading term S_1^a reveals that, once again, a partial resummation of higher powers of the same graphs is possible: one finds that

$$-S_{\text{eff}}^a = \log \left[\prod_x \det \left(1 + h_1 W_x \right)^{2N_f} \det \left(1 + \bar{h}_1 W_x^\dagger \right)^{2N_f} \right] + \dots ; \quad h_1 = (2\kappa e^{a\mu})^{N_\tau} (1 + \dots) , \quad (7)$$

The expression for h_1 received also contributions from various types of higher-order graphs; including these, and the first subleading fermionic term S_2^a , one can write the model as [6, 7]:

$$Z_{\text{eff}} = \int [\mathcal{D}L] \prod_{\langle ij \rangle} \left(1 + 2\lambda \text{Re} L_i L_j^* \right) \prod_x \det \left[(1 + h_1 W_x)(1 + \bar{h}_1 W_x^\dagger) \right]^{2N_f} \\ \prod_{\langle ij \rangle} \left[1 - h_2 \text{Tr} \frac{W_i}{1 + C W_i} \text{Tr} \frac{W_j}{1 + C W_j} \right] \left[1 - \bar{h}_2 \text{Tr} \frac{W_i^\dagger}{1 + \bar{C} W_i^\dagger} \text{Tr} \frac{W_j^\dagger}{1 + \bar{C} W_j^\dagger} \right] , \quad (8)$$

(we rename $\lambda_1 \rightarrow \lambda$ and ignore higher-order corrections). Despite the appearance of W_x and W_x^\dagger in the above, Z_{eff} can still be expressed, in practice, in terms only of the θ, ϕ parameterising L_x in Eq. (1). The effective couplings are given by:

$$h_1 = C \left(1 + 6\kappa^2 N_\tau \frac{u - u^{N_\tau}}{1 - u} + \dots \right) ; \quad (9)$$

$$h_2 = C^2 \frac{\kappa^2}{3} \left(1 + 2 \frac{u - u^{N_\tau}}{1 - u} \right) ; \quad (10)$$

$$C = (2\kappa e^{a\mu})^{N_\tau} . \quad (11)$$

⁵ The next term seems an ordinary nearest-neighbour interaction, but is actually non-centre-symmetric, Eq. (8).

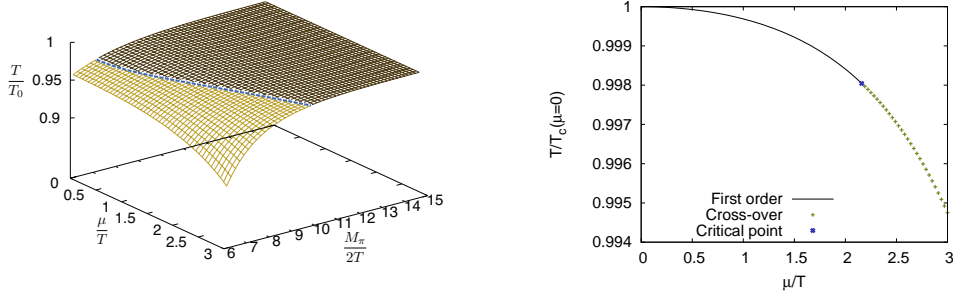


Figure 1. *Left:* $N_f = 2$ phase diagram in the space of temperature, chemical potential and meson mass (here called M_π). T_0 is the pure-gauge transition point; the blue line is the critical line. *Right:* T - μ phase diagram for the choice $N_f = 2$, $M_{ud}/T \simeq 8.68$; the vertical axis is normalised to the $\mu = 0$ transition temperature for this mass choice.

The meson and baryon masses are also evaluated:

$$am_M = -2 \ln(2\kappa) - 6\kappa^2 - 24\kappa^2 \frac{u}{1-u} + \dots ; \quad am_B = -3 \ln(2\kappa) - 18\kappa^2 \frac{u}{1-u} + \dots \quad (12)$$

The model, as formulated in Eq. (8), is well suited to different simulational strategies: we could confirm that the standard Metropolis approach, a flux-representation-based worm algorithm, and a complex-Langevin implementation all agree with each other: however, one has to keep in mind that the first has to rely on a reweighting procedure for $\mu > 0$ (which does not hinder its scope significantly for not too large spatial volumes) and the second ceases to be of use if the h_2 term is included, while the third does not have a sign problem at all and is thus the best choice to deal with large-volume, finite- μ ensembles. All of these algorithms, when compared to traditional lattice QCD approaches, are far less expensive in terms of required system resources and CPU time.

An important application of this effective approach concerns the mapping of the critical surface in the $(M_{ud}, M_s, \mu/T)$ space. Due to the heavy-quark validity of the model, the second-order surface that could be located is the one associated to the upper-right corner of the Columbia plot: there, a nice agreement between the findings at zero, real positive and imaginary μ was verified and the correct universality class of the transition was confirmed. Moreover, the result at zero chemical potential for $N_f = 1, 2, 3$ closely reproduces those found in ordinary QCD simulations e.g. in [8, 9]. The shape of the critical surface at all chemical potentials can be summarised in the parametrisation [6]:

$$(N_f h_1 e^{-\mu/T})_{\text{crit}} = \frac{0.00075(1)}{\cosh \mu/T} ; \quad \kappa_f \simeq e^{-aM_f/2}, \quad M_f \rightarrow \infty ; \quad (13)$$

the phase diagram for heavy-quark QCD can then be determined in $(\frac{\mu}{T}, \frac{M_\pi}{2T}, \frac{T}{T_0})$, and, by taking slices of it, a “heavy fermion” version of the familiar μ - T phase diagram can be drawn (Fig. 1).

3. Low-temperature regime

We now turn to investigate the cold and dense regime of QCD with this effective theory. In the (analytically solvable) static strong-coupling limit, one finds that the Silver Blaze property holds as $T \rightarrow 0$, with the quark number density approaching a step function centred at $3\mu = m_B = -3 \log(2\kappa)$ (the last equality is exact in this limit):

$$a^3 n \xrightarrow{T \rightarrow 0} 2N_c \Theta(\mu_B - m_B) ; \quad (14)$$

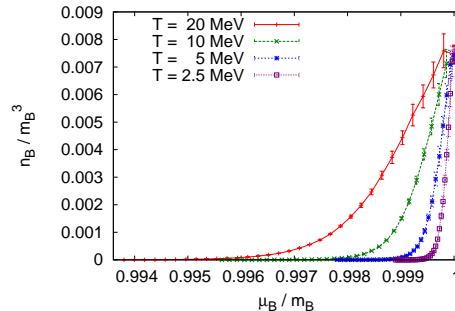


Figure 2. Continuum-extrapolated baryon density, in units of m_B^3 , as a function of the baryon chemical potential (also measured in units of m_B) for various temperatures.

one sees then also that a saturation phenomenon is expected (this is connected to the resummations performed in deriving the form of Z_{eff}). The next step is to perform numerical computations with the full expression Eq. (8). Taking advantage of the fact that N_τ is now merely a parameter in the effective couplings' maps, we can tune it to very large values in the hundreds: this effectively realises $\lambda \rightarrow 0$ thus simplifying the model; by keeping $\kappa \sim 10^{-3}$, moreover, we can still trust the pure-Yang-Mills scale-setting prescription [10]. In this way, we can measure numerically the baryon density as a function of the baryon chemical potential with $m_B = 30$ GeV and four different temperatures from 20 MeV to 2.5 MeV [7].

The baryon density n_B can be expressed as a meaningful physical quantity after a careful continuum limit is taken: we could generate data at nine different values of the lattice spacings a and perform a fit to the continuum limit; n_b is measured as

$$a^3 n_b = \frac{a^3}{3} \left(\frac{T}{V} \frac{\partial \log Z}{\partial \mu} \right) = - \frac{1}{3N_s^3} \left\langle \frac{\partial S_{\text{eff}}}{\partial (\mu/T)} \right\rangle. \quad (15)$$

The results are shown in Fig. 2: the step-like shape of the density is less and less smoothed-out as $T \rightarrow 0$ (and anyway the μ -scale is extremely tiny); besides, as the baryonic chemical potential hits the baryon mass the expected saturation to a common, roughly T -independent density is seen, and this value, when expressed in appropriate units of m_B , is less than a factor two off the real-world nuclear matter, despite this model being limited to extremely massive quarks. This seems to suggest that the core features of nuclear condensation might be largely independent of whether the fundamental constituents of the baryon are heavy or light.

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